Journal of Applied Mechanics and Technical Physics, Vol. 38, No. 5, 1997

EXACT SOLUTIONS OF THE EQUATIONS OF VORTEX SHALLOW WATER

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UDC 532.591+517.958

Exact solutions of the equations of vortex shallow water were derived by Freeman [1] for simple waves. Sachdev and Varugheze [2] found a group of transformations that are admitted by a system of differential equations and some exact solutions. Stationary solutions with backward-flow regions were obtained by Varley and Blythe [3]. Teshukov [4] formulated hyperbolicity conditions for a system of equations of motion by generalizing this notion for a class of integro-differential equations proposed in [5].

The present paper is devoted to the construction of exact solutions of the integro-differential equations of vortex shallow water, which describe vortex flows of an ideal incompressible fluid with a free boundary in a gravity field in the Euler–Lagrangian coordinate system. The simpler subsystems that determine the classes of exact solutions are given using a group of transformations that are admitted by the system of equations considered, and some of these subsystems are integrated. Solutions that describe the fluid motion with the formation of backward-flow regions are found in the unsteady-state case. The solution on which a system of integro-differential equations loses its hyperbolicity with time is given.

1. The Model of Vortex Shallow Water and Admissible Transformations. The solution of the boundary-value problem

$$u_T + uu_X + vu_Y + h_X = 0, \quad u_X + v_Y = 0, \quad h_T + u(T, X, h)h_X = v(T, X, h), \quad v(T, X, 0) = 0 \qquad (1.1)$$

describes, in the $-\infty < X < \infty$, $0 \leqslant Y \leqslant h(T, X)$ region, plane-parallel eddy motions of a layer of an ideal incompressible fluid with a free boundary $Y = h(T, X)$ in a gravity field in a shallow-water approximation [6]. System (1.1) is derived as a result of the nondimensionalization of the Euler equations and taking into account the fact that, for the flows considered, the ratio of the characteristic vertical scale to the horizontal one is small.

As is shown by Zakharov [7], with the change of variables

$$T = t, \quad X = x, \quad Y = \Phi(t, x, \lambda) \qquad (0 \leqslant \lambda \leqslant 1), \qquad (1.2)$$

where the function $\Phi(t, x, \lambda)$ is a solution of the Cauchy problem $\Phi_t + u(t, x, \Phi)\Phi_x = v(t, x, \Phi)$ and $\Phi(0, x, \lambda) = \Phi_0(x, \lambda)$, one can map the flow region onto the band $0 \leqslant \lambda \leqslant 1$, $-\infty < X < \infty$, and the functions $u(t, x, \lambda)$ and $H(t, x, \lambda) = \Phi_\lambda$ are defined from the system

$$u_t + uu_x + \int_0^1 H_x \, d\lambda = 0, \qquad H_t + Hu_x + uH_x = 0. \qquad (1.3)$$

The change of variables (1.2) is invertible under the condition that $\Phi_\lambda \neq 0$. Below, we assume that $\Phi_\lambda > 0$ [$\Phi(t, x, 0) = 0$ and $\Phi(t, x, 1) = h(t, x)$]. In the absence of vorticity ($\omega = -u_Y = -u_\lambda H^{-1} = 0$), the model (1.3) is reduced to the known shallow-water equations.

It is easy to see that system (1.3) is invariant under the transformation group G_5: (1) $t' = t + a$, (2) $x' = x + a$, (3) $x' = at + x$ and $u' = u + a$, (4) $t' = at$ and $x' = ax$, and (5) $x' = ax$, $u' = au$, and $H' = a^2 H$. Applying the method proposed by Ovsyannikov [8], one can construct invariant solutions based on the Lie algebra L5 of the following operators: $X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = t\partial_x + \partial_u$, $X_4 = t\partial_t + x\partial_x$, and $X_5 = x\partial_x + u\partial_u + 2H\partial_H$, which correspond to the transformations (1)–(5).

Novosibirsk State University, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 38, No. 5, pp. 44–55, September–October, 1997. Original article submitted December 18, 1995; revision submitted March 25, 1996.

 0021-8944/97/3805-0692 $18.00

For rational use of the available transformations, in finding invariant solutions, we give the optimal system of subalgebras of the Lie algebra of operators L_5, which are constructed using the algorithm given in [9]. All representatives of the optimal system of rank-1 subalgebras are as follows: (1) αX_4+X_5, (2) $X_2-X_4+X_5$, (3) X_1+X_5, (4) X_3+X_4, (5) X_4, (6) X_1+X_3, (7) X_3, (8) X_2, and (9) X_1. The system is optimal in the sense that the solutions obtained by means of its representatives give all possible invariant solutions, which correspond to one-parametric subgroups of the transformation group G_5 up to the change of variables. The subsequent construction of invariant solutions is reduced to finding the invariants of the corresponding subalgebra and also to the integration of the factor-systems obtained.

2. Systems of Equations Determining Invariant Solutions. For all representatives of the rank-1 optimal system, we give a set of basic invariants J, the representation of the solution, and the factor-system E/H, where $H(\alpha^i X_i)$ denotes the subalgebra.

(1) $H(\alpha X_4+X_5)$ and $J = (xt^{-(1+1/\alpha)},\ \lambda,\ ut^{-1/\alpha}, Ht^{-2/\alpha})$. The solutions are invariant under extensions of all α-dependent variables ($\alpha \neq 0$). They describe the class of self-similar (in the narrow sense) fluid motions. The solution is represented as follows: $u = t^{\beta}\varphi(\xi,\lambda)$, $H = t^{2\beta}\psi(\xi,\lambda)$, $\xi = xt^{-(1+\beta)}$, and $\beta = \alpha^{-1}$. The factor-system E/H is

$$\beta\varphi - (1+\beta)\xi\varphi' + \varphi\varphi' + \int_0^1 \psi'\,d\lambda = 0, \quad 2\beta\psi - (1+\beta)\xi\psi' + (\varphi\psi)' = 0, \tag{2.1}$$

where differentiation is performed with respect to the variable ξ.

For $\alpha = 0$, we have $J = (t, \lambda, ux^{-1}, Hx^{-2})$. The solutions are invariant under extensions of x, u, and H. The representation of the solution is as follows: $u = x\varphi(t,\lambda)$ and $H = x^2\psi(t,\lambda)$. The factor-system E/H is

$$\varphi_t + \varphi^2 + 2\int_0^1 \psi\,d\lambda = 0, \quad \psi_t + 3\varphi\psi = 0. \tag{2.2}$$

(2) $H(X_2 - X_4 + X_5)$ and $J = (t\exp(x), \lambda, tu, t^2H)$. The solutions are invariant under simultaneous displacements in the direction of the x axis, time dilatation, and extensions of u and H. The solution is represented in the following form: $u = t^{-1}\varphi(\xi,\lambda)$, $H = t^{-2}\psi(\xi,\lambda)$, and $\xi = t\exp(x)$. The factor-system E/H is

$$(\varphi + \varphi^2/2 + \int_0^1 \psi\,d\lambda)' = \varphi/\xi, \quad (\psi + \varphi\psi)' = 2\psi/\xi. \tag{2.3}$$

(3) $H(X_1 + X_5)$ and $J = (x\exp(-t), \lambda, u\exp(-t), H\exp(-2t))$. The solutions are invariant under simultaneous displacements with respect to t and under extensions of x, u, and H. The solution is represented as follows: $u = \varphi(\xi,\lambda)\exp(t)$, $H = \psi(\xi,\lambda)\exp(2t)$, and $\xi = x\exp(-t)$. The factor-system E/H is

$$\varphi - \xi\varphi' + \varphi\varphi' + \int_0^1 \psi'\,d\lambda = 0, \quad 2\psi - \xi\psi' + (\varphi\psi)' = 0. \tag{2.4}$$

(4) $H(X_3 + X_4)$ and $J = (t^{-1}\exp(xt^{-1}), \lambda, t^{-1}\exp(u), H)$. The solutions are invariant under Galilei simultaneous transformations along the x axis and under uniform extensions of t and x. The representation of the solution is as follows: $u = \ln(t\varphi(\xi,\lambda))$, $H = \psi(\xi,\lambda)$, and $\xi = t^{-1}\exp(xt^{-1})$. The factor-system E/H is

$$\varphi + \xi\varphi'(\ln(\varphi\xi^{-1}) - 1) + \xi\varphi\int_0^1 \psi'\,d\lambda = 0, \quad \varphi'\psi + \varphi\psi'(\ln(\varphi\xi^{-1}) - 1) = 0. \tag{2.5}$$

(5) $H(X_4)$ and $J = (xt^{-1}, \lambda, u, H)$. The solutions are invariant under uniform extensions of the variables t and x. They describe the class of self-similar fluid motions. The representation of the solution is as follows:

$u = \varphi(\xi, \lambda)$, $H = \psi(\xi, \lambda)$, and $\xi = xt^{-1}$. The factor-system E/H is

$$-\xi\varphi' + \varphi\varphi' + \int_0^1 \psi'\, d\lambda = 0, \quad -\xi\psi' + (\varphi\psi)' = 0. \tag{2.6}$$

(6) $H(X_1 + X_3)$ and $J = (x - t^2/2, \lambda, u - t, H)$. The solutions are invariant under simultaneous displacements with respect to t and under Galilei transformations. The solution is represented as follows: $u = \varphi(\xi, \lambda) + t$, $H = \psi(\xi, \lambda)$, and $\xi = x - t^2/2$. The factor-system E/H is

$$1 + \varphi\varphi' + \int_0^1 \psi'\, d\lambda = 0, \quad (\varphi\psi)' = 0. \tag{2.7}$$

(7) $H(X_3)$ and $J = (t, \lambda, x - tu, H)$. The solutions are invariant under Galilei transformations. They describe the class of Galilei-invariant solutions. The representation of the solution is as follows: $u = (x - \varphi(t, \lambda))t^{-1}$ and $H = \psi(t, \lambda)$. The factor-system E/H is

$$\varphi_t = 0, \quad \psi_t + \psi t^{-1} = 0. \tag{2.8}$$

(8) $H(X_2)$ and $J = (t, \lambda, u, H)$. The solutions are invariant under displacements on the x axis. The representation of the solution is $u = \varphi(t, \lambda)$ and $H = \psi(t, \lambda)$. The factor-system E/H is

$$\varphi_t = 0, \qquad \psi_t = 0. \tag{2.9}$$

(9) $H(X_1)$ and $J = (x, \lambda, u, H)$. The solutions are invariant under time dilatations. They describe the class of stationary fluid motions. The representation of the solution is $u = \varphi(x, \lambda)$ and $H = \psi(x, \lambda)$. The factor-system E/H is

$$\varphi\varphi_x + \int_0^1 \psi_x\, d\lambda = 0, \quad (\varphi\psi)_x = 0. \tag{2.10}$$

In all the cases enumerated above, one can reduce the factor-systems to a single equation. In what follows, the change of variables and the forms of the resulting integro-differential equations are given.

(1) Making a change of the dependent variables

$$w(\xi, \lambda) = \exp\Big(\int_0^\xi ((1+\beta)s - \varphi(s, \lambda))^{-1}\, ds\Big),$$

one reduces the factor-system (2.1) for $\beta \neq -3^{-1}$ to the equation

$$w^2 w'' + 2\beta w(w')^2 - \beta(1+\beta)\xi(w')^3 - (w')^3 \int_0^1 C(\lambda) w^{3\beta - 1}(3\beta(w')^2 + ww'')\, d\lambda = 0.$$

The functions φ and ψ that determine the invariant solution are expressed via w according to the formulas $\varphi = (1+\beta)\xi - w/w'$ and $\psi = C(\lambda)w^{3\beta}w'$, where $C(\lambda)$ is an arbitrary function.

In the case $\beta = -3^{-1}$, to define the function $w = \varphi - (2/3)\xi$, we obtain the equation

$$ww' + 3^{-1}w - (2/9)\xi - \int_0^1 C(\lambda) w^{-2} w'\, d\lambda = 0.$$

The functions φ and ψ are expressed via w by the formulas $\varphi = w + (2/3)\xi$ and $\psi = C(\lambda)w^{-1}$.

Let $w(t, \lambda) = \exp\Big(\int_0^t \varphi(s, \lambda)\, ds\Big)$ be a new desired function. The factor-system (2.2) is then reduced to the integro-differential equation

$$w_{tt} + 2w \int_0^1 C(\lambda) w^{-3}\, d\lambda = 0, \tag{2.11}$$

while the functions φ and ψ are related to w by the formulas $\varphi = w_t w^{-1}$ and $\psi = C(\lambda) w^{-3}$.

(2) Changing the independent and dependent variables

$$\tau = \ln \xi, \qquad w(\tau, \lambda) = \exp\Big(\int_0^{\tau} (\varphi(s, \lambda) + 1)^{-1}\, ds \Big),$$

one reduces the factor-system (2.3) to the equation

$$w^2 w_{\tau\tau} - w_\tau^3 - w_\tau^3 \int_0^1 C(\lambda)(w w_{\tau\tau} + w_\tau^2)\, d\lambda = 0. \tag{2.12}$$

The functions φ and ψ that determine the invariant solution are expressea via w by the formulas $\varphi = w w_\tau^{-1} - 1$ and $\psi = C(\lambda) w w_\tau$.

(3) Changing the dependent variable $w(\xi, \lambda) = \exp\Big(\int_0^{\xi} (s - \varphi(s, \lambda))^{-1}\, ds \Big)$, one reduces the factor-system (2.4) to the equation

$$w^2 w'' + w(w')^2 - (w')^3 \xi - (w')^3 \int_0^1 C(\lambda) w (w w'' + 2(w')^2)\, d\lambda = 0.$$

The functions φ and ψ are expressed via w by the formulas $\varphi = \xi - w/w'$ and $\psi = C(\lambda) w^2 w'$.

(4) Changing the dependent variable

$$w(\xi, \lambda) = \exp\Big(-\int_0^{\xi} (s \ln(\varphi(s, \lambda) s^{-1}) - 1)^{-1}\, ds \Big),$$

one reduces the factor-system (2.5) to the equation

$$\xi w^2 w'' + w^2 w' - \xi^2 (w')^3 + \xi^3 (w')^3 \int_0^1 C(\lambda)(w' + \xi w'')\, d\lambda = 0.$$

The functions φ and ψ are expressed through w by the formulas $\varphi = \xi \exp(1 - w(\xi w')^{-1})$ and $\psi = -C(\lambda) \xi w'$.

(5) Changing the dependent variable $w(\xi, \lambda) = \exp\Big(\int_0^{\xi} (s - \varphi(s, \lambda))^{-1}\, ds \Big)$, one reduces the factor-system (2.6) to the equation

$$w^2 w'' - (w')^3 \int_0^1 C(\lambda) w''\, d\lambda = 0.$$

The functions φ and ψ are expressed via w by the formulas $\varphi = \xi - w/w'$ and $\psi = C(\lambda) w'$.

3. Invariant Solutions. The results of integration of some factor-systems are given, and the solutions found are analyzed.

(A) Subsystems (2.1). For $\beta = -2^{-1}$, system (2.1) is integrated, and the solution is of the form

$$\varphi(\xi, \lambda) = 2^{-1}\xi \pm \sqrt{4^{-1}\xi^2 + C(\lambda) - 2\eta(\xi)}, \quad \psi(\xi, \lambda) = C'(\lambda)(D(\lambda)(\xi - 2\varphi))^{-1} \exp\Big(-\int_0^{\xi} (\tau - 2\varphi)^{-1}\, d\tau \Big).$$

Here $C(\lambda)$ and $D(\lambda)$ are arbitrary functions, and $\eta(\xi)$ is found from the equation $\eta = \int_0^1 \psi\,d\lambda$. We consider the flow region, where $\varphi - 2^{-1}\xi < 0$. Let $D(\lambda) \equiv 1$, $C'(\lambda) > 0$, $C(1) = C_1$, and $C(0) = C_0$. For definition of the function η, we obtain the equation

$$\eta(\xi) = \int_{C_0}^{C_1} (\xi^2 + 4C - 8\eta(\xi))^{-1/2} \exp\Big(-\int_0^{\xi} (\tau^2 + 4C - 8\eta(\tau))^{-1/2}\,d\tau\Big)\,dC,$$

whose solution can be found by the method of successive approximations. The function η_i is defined by the substitution of the function η_{i-1} calculated at the previous step into the right-hand side of the equation considered. The approximation $\eta_0 = \eta(0)$, which is found from the equation $\sqrt{C_1 - 2\eta_0} - \sqrt{C_0 - 2\eta_0} = \eta_0$, is chosen as the initial one. After the inequality $2\sqrt{C_1 - C_0 + 4} \leqslant C_0$ is satisfied, the iteration process converges for any $\xi \geqslant 0$.

Under the assumption that $\int_0^1 \psi\,d\lambda = l\xi^2$, $0 < l = \text{const} < 9^{-1}$, and $-1/3 < \beta \leqslant 0$, integration of the factor-system (2.1) produces the solution given in [10].

In the case $l = 9^{-1}$ and $\beta = -2/3$, we have the solution

$$u = 2x(3t)^{-1} + 3^{-1}f_1^{-1}(\lambda)f_2(\lambda)t^{-2/3},\quad H = -3(f_1^2(\lambda)f_2^{-1}(\lambda)x^2t^{-2} + 2f_1(\lambda)xt^{-5/3} + f_2(\lambda)t^{-4/3}),$$

$$\int_0^1 f_j(\lambda)\,d\lambda = 0 \qquad (j = 1,\ 2), \qquad \int_0^1 f_1^2(\lambda)f_2^{-1}(\lambda)\,d\lambda = -27^{-1}.$$

The flow is vortex, and the free-boundary equation is $y = x^2(3t)^{-2}$.

(B) Subsystem (2.2). We search for the solution of Eq. (2.11) in the form $w = a(\lambda)g_1(t) + b(\lambda)g_2(t)$. It is assumed that $d'(\lambda) = C(\lambda)(a(\lambda))^{-3} > 0$, $d(\lambda) = b(\lambda)(a(\lambda))^{-1}$, $d_1 = d(1)$, and $d_0 = d(0)$. As a result of integration of Eq. (2.11) and transition to the initial functions φ and ψ, we find their parametric representations

$$\begin{gathered}\varphi = \tilde{\varphi}(\tau,\lambda) = \frac{(d_1 - d(\lambda))k_1F^2(\tau)}{d(\lambda) - d_0 + (d_1 - d(\lambda))(k_1\tau + k_2)} - F(\tau)F'(\tau),\\ \psi = \tilde{\psi}(\tau,\lambda) = \frac{(d_1 - d_0)^3 d'(\lambda)F^3(\tau)}{[d(\lambda) - d_0 + (d_1 - d(\lambda))(k_1\tau + k_2)]^3}, \qquad t = \int_0^{\tau} F^{-2}(\tau')\,d\tau', \quad \tau \geqslant 0.\end{gathered} \tag{3.1}$$

Here $F(\tau) = (d_1 - d_0)k_1^{-2}[(k_1\tau + k_2 - 1)\ln(k_1\tau + k_2) - (k_1\tau + k_2)] + k_3\tau + k_4$, $k_1 > 0$, $k_2 > 0$, and k_3 and k_4 are the integration constants.

The function $F(\tau)$ is determined for $\tau > 0$ and is concave. The inequality $H > 0$ is fulfilled if $F(\tau) > 0$. The free-boundary equation $y = h(t,x) = \tilde{h}(\tau,x)$ is given by the formula

$$\tilde{h}(\tau,x) = 2^{-1}(d_1 - d_0)(F(\tau))^3(1 + (k_1\tau + k_2)^{-1})(k_1\tau + k_2)^{-1}x^2 \tag{3.2}$$

and is shaped like a parabola with branches directed upward at each fixed moment of time. Analysis shows that the initial data determine one of three possible flow regimes. For $t = 0$, the free boundary is specified by the formula $y = lx^2$ (the constant $l > 0$ is determined by the initial data).

Regime No. 1. $F(\tau) > 0$, $F'(\tau) < 0$, and $F(\tau_1) = 0$. The parameter τ varies from 0 to τ_1, and the time t varies from zero to infinity. By virtue of (3.2) and the behavior of the function F, the depth falls off to zero for $t = \infty$ with time. The horizontal velocity component $u = x\tilde{\varphi}$ is positive for $x > 0$ and negative for $x < 0$ ($\tau < \tau_1$), and, hence, the fluid outflows at infinity.

Regime No. 2. $F(\tau) > 0$ and $F'(\tau) > 0$. The time t varies from 0 to $t_1 = \int_0^{\infty} F^{-2}(\tau) < \infty$, because the integral converges. For finite time, the depth becomes infinite everywhere, except for the point $x = 0$, which occurs owing to fluid inflow from infinity.

Regime No. 3. $F(\tau) > 0$, $F'(\tau) < 0$ $(0 \leqslant \tau < \tau_0)$, and $F'(\tau) > 0$ $(\tau_0 < \tau < \tau_1)$. The time t varies from 0 to $t_1 < \infty$. This case is a combination of the two previous ones. Regime No. 1 is realized before the definite moment $t_0 = t(\tau_0) < t_1$, and regime No. 2 occurs for $t > t_0$.

The solution considered admits the following: $u = x\varphi(t,\lambda) + \varphi_1(t,\lambda)$ and $H = x^2\psi(t,\lambda) + x\psi_1(t,\lambda) + \psi_2(t,\lambda)$. The functions φ and ψ are found from system (2.2), and for determination of the functions φ_1, ψ_1, and ψ_2, we obtain the equations

$$(\varphi_1)_t + \varphi\varphi_1 + \int_0^1 \psi_1\, d\lambda = 0, \quad (\psi_1)_t + 2\varphi\psi_1 + 2\varphi_1\psi = 0, \quad (\psi_2)_t + \varphi_1\psi_1 + \varphi\psi_2 = 0.$$

We give the following particular solutions:

(1) The functions φ and ψ are given by formulas (3.1),

$$\varphi_1 = \psi_1 = 0, \quad \psi_2(t,\lambda) = l(\lambda)\exp\Big(-\int_0^t \varphi(t',\lambda)\, dt'\Big), \quad l(\lambda) > 0;$$

this solution is similar to that considered above, but the depth at the point $x = 0$ is greater than zero;

(2) Assuming the functions ψ and ψ_1 be equal to zero, we find the solution

$$u = (x + C_2(\lambda))(t + C_1(\lambda))^{-1}, \qquad H = C_3(\lambda)(t + C_1(\lambda))^{-1}. \tag{3.3}$$

Let the initial data for system (1.3) be of the form

$$u(0,x,\lambda) = a(\lambda)x + b(\lambda),\ H(0,x,\lambda) = H_0(\lambda)\ \Big[a(\lambda) \neq 0,\ H_0(\lambda) > 0,\ \int_0^1 H_0(\lambda)\, d\lambda = h_0 < \infty\Big].$$

Formula (3.3) with the functions $C_1(\lambda) = a^{-1}(\lambda)$, $C_2(\lambda) = b(\lambda)a^{-1}(\lambda)$, and $C_3(\lambda) = H_0(\lambda)a^{-1}(\lambda)$ gives the solution of this Cauchy problem. If $a(\lambda) > 0$, the solution is determined for all $t \geqslant 0$ and describes the compression of a liquid strip under the action of pressure. The depth $h(t)$ decreases with time from h_0 at $t = 0$ to zero at $t = \infty$. The horizontal velocity component is positive for $x + b(\lambda)a^{-1}(\lambda) > 0$ and negative for $x + b(\lambda)a^{-1}(\lambda) < 0$, and, hence, the fluid outflows to infinity. In the case $a(\lambda) < 0$, the solution is determined for $t \in [0, M]$, $M = -\min a^{-1}(\lambda)$, and describes the reverse process. The depth increases with time owing to fluid inflow from infinity. At time $t = M$, the depth can be finite or infinite, depending on the initial distribution.

(C) Subsystem (2.3). Let us consider Eq. (2.12). It is easy to see that the function $w(\tau,\lambda) = a(\lambda)\tau + b(\lambda)$ is a solution if $1 + \int_0^1 C(\lambda)(a(\lambda))^2\, d\lambda = 0$. In going to the initial functions φ and ψ and assuming that $C(\lambda) = (a(\lambda))^{-2}$, we obtain the following invariant solution:

$$u = (x + \ln t + d(\lambda) - 1)t^{-1},\ H = -(x + \ln t + d(\lambda))t^{-2}. \tag{3.4}$$

Solution (3.4) is determined for $x < -M - \ln t$, where $M = \max d(\lambda)$. The free boundary is shaped like a straight line with the angular coefficient $-t^{-2}$. In the solution domain, the function $u(t,x,\lambda) < 0$.

(D) Subsystem (2.9). The factor-system (2.9) describes the class of steady-state vortex flows. They were analyzed comprehensively by Varley and Blythe [3].

(E) Subsystem (2.10). The solution of the factor-system (2.10) $u = u(\lambda)$ and $H = H(\lambda)$ describes shear flows. In the Euler coordinate system, the solution is of the form $u = u(y)$, $v = 0$, and $h = \text{const}$.

(F) The invariant solution $u = t$, $H = (x - t^2/2)a(\lambda)$, and $\int_0^1 a(\lambda)\, d\lambda = -1$ is constructed using the subalgebra $X_1 + X_3$ and $X_4 + X_5$. In the Euler coordinates, it is of the form $u = t$, $v = 0$, and $h = t^2/2 - x$. Let $x = t^2/2 - h_0$ be a rigid wall (h_0 is a positive constant); $u = x'(t) = t$ and $h = h_0$ on it; the function $h(t,x) = 0$ at the point $x = t^2/2$, and, hence, the fluid is in the triangle $t^2/2 - h_0 \leqslant x \leqslant t^2/2$ $0 \leqslant y \leqslant t^2/2 - x$

at each moment of time and its velocity relative to the coordinate system moving with velocity t is equal to zero. The flow considered can be interpreted as the uniformly accelerated motion of a liquid wedge at rest. This solution admits a generalization. The following formulas describe a similar vortex flow:

$$u = t + f(\lambda), \quad H = (x - t^2/2)a(\lambda) - tf(\lambda)a(\lambda) + g(\lambda), \quad \int_0^1 a(\lambda)\,d\lambda = -1.$$

Unsteady-State Flows with a Critical Layer. We use here the methods developed by Varley and Blythe [3] in a study of flows with a critical layer and obtain a class of exact solutions that describe nonstationary fluid motions, with the formation of backward-flow regions.

As a result of integration of the factor-system (2.7), we find the solution

$$u = t \pm \sqrt{2(C(\lambda) - \xi - h)}, \quad H = D(\lambda)(u - t)^{-1}, \quad \xi = x - t^2/2, \tag{3.5}$$

where the function $h(\xi)$ is defined from the equation $F = h - \int_0^1 H\,d\lambda = 0$.

Sign alteration occurs for $C(\lambda) = \xi + h(\xi)$. We consider the flow region in which $(u - t)$ does not change sign. In what follows, we assume that $D(\lambda) \equiv -1$ and $C(\lambda)$ is a strictly increasing and continuously differentiable function. We use the minus sign in the formula that expresses velocity. We denote the minimum value of C_m by $C(\lambda)$. In this case, the equation for determination of $h(\xi)$ is of the form

$$F(h, \xi) = h - \int_0^1 (2(C(\lambda) - \xi - h))^{-1/2}\,d\lambda = 0. \tag{3.6}$$

Let us analyze Eq. (3.6) from a qualitative point of view. The function $F(h,\xi)$ is defined in the domain $h+\xi \leqslant C_m$ $(0 \leqslant h < \infty)$. For $h+\xi < C_m$, there are arbitrary continuous functions $F(h,\xi)$, the derivative with respect to ξ and the secondary derivative being negative. There exists a unique value $\xi_0 \in (-\infty, C_m - h_0]$ such that $F(h_0, \xi_0) = 0$, because $F(h_0, \xi) \to h_0 > 0$ for $\xi \to -\infty$ and $F(h_0, \xi) \to h_0 - b \leqslant 0$ for $\xi \to C_m - h_0$ and $F_\xi < 0$ in the cross sections $h = h_0$, $h_0 \in (0, b]$, where $b = \int_0^1 (2(C(\lambda) - C_m))^{-1/2}\,d\lambda$ [the integral converges by virtue of the conditions to which the function $C(\lambda)$ is subject]. The equation $F(h, \xi_0) = 0$ has a single root in the cross sections $\xi = \xi_0$ $[\xi_0 \in (-\infty, a)]$ and $a = C_m - b$, because $F(0, \xi_0) < 0$ and $F(C_m - \xi_0, \xi_0) = a - \xi_0 > 0$, and $F_h(h, \xi_0)$ can change sign only once $(F_{hh} < 0)$.

Now let us consider the function $F(h, \xi)$ in the cross sections $\xi = \xi_0 \geqslant a$. The function F vanishes twice with variation of h from 0 to b in the cross section $\xi = a$. Indeed, $F(b, a) = 0$ and $F_h \to -\infty$ for $h \to b$. This means that for some h, which are close enough to b, we have $F(h, a) > 0$, $F(0, a) < 0$. Thus, there is a value of h_0 $(0 < h_0 < b)$ such that $F(h_0, a) = 0$ (the value of h_0 is unique owing to the convexity of F over h). There is no difficulty in seeing that for the ξ_0 values close to a, the function $F(h, \xi_0)$ vanishes twice when h ranges from 0 to $C_m - \xi_0$. We denote the maximum value of ξ_0 at which the equation $F(h, \xi_0) = 0$ has a root by d $(a < d < C_m)$. Let h_0 be the root of the equation $F(h, d) = 0$. At this point, we then have

$$F_h = 1 - \int_0^1 H(u - t)^{-2}\,d\lambda = 0,$$

and, according to [5], the curves $x - (1/2)t^2 = \xi = \text{const}$ are the characteristics.

Figure 1 shows the curve $F(h, \xi) = 0$, which corresponds to the function $C(\lambda) = \lambda + 3$. In this case, $C_m = 3$, $d \approx 1.885$, $a = 3 - \sqrt{2}$, and $b = \sqrt{2}$. For the initial coordinate $x = \xi + t^2/2$, the curve in Fig. 1 corresponds to time $t = 0$. At any other moment of time $t = t_0$, the diagram of the equation is obtained by shifting the quantity $t_0^2/2$ to the right along the x axis, which corresponds to the uniformly accelerated motion. The derivative of the function $h(\xi)$ is of the form $h' = (1 - F_h)F_h^{-1}$. With variation in the function $C(\lambda)$, the qualitative behavior of this curve remains.

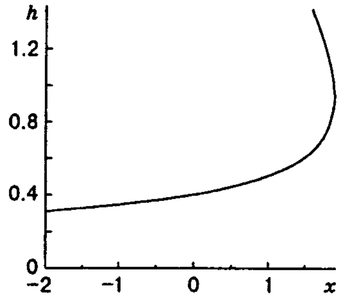

Fig. 1

The foregoing considerations show that the equation $F(h,\xi) = 0$ has two branches of solutions for $\xi < d$. The lower branch $h_1(\xi)$ is determined for all $\xi < d$. Formulas (3.5) with a free boundary $h = h_1(\xi)$ specify an invariant solution in this interval; in the flow region, $(u-t)$ does not change sign (there is no critical layer). The upper branch $h_2(\xi)$ is determined for $\xi \in [a,d]$. Let us consider the problem of the extension of solution (3.5) with the function $h = h_2(\xi)$ to the $\xi < a$ region. As a result, we construct a solution that describes the nonstationary flow with a critical layer. For $\xi \leqslant a$, we specify the free-boundary equation $h_2(\xi)$ arbitrarily and require satisfaction of the following conditions: (1) $h_2(a) = b$; (2) $h_2'(a) = -1$; (3) $h_2'(\xi) > -1$ and $h_2(\xi) > h_1(\xi)$ $(\xi < a)$; (4) $h_2(\xi) \to 0$ and $h_2'(\xi) \to 0$ $(\xi \to -\infty)$.

For $\xi \leqslant a$, we specify the upper boundary of the backward-flow region by the equation $y = g(\xi)$, where

$$g(\xi) = h_2(\xi) - \int_0^1 (2(C(\lambda) - \xi - h_2(\xi)))^{-1/2}\, d\lambda, \tag{3.7}$$

the lower boundary of this region being $y = 0$. In the region $0 \leqslant y \leqslant g(\xi)$, we construct the flow possessing the following property: in a definite curve lying in this region, the function u changes sign, and the streamlines in the coordinate system moving with velocity t are arranged as shown in Fig. 2. For $\xi < a$, we determine the solution by formulas (3.5) with the given function $h = h_2(\xi)$ in the outer region (from the boundary of the backward-flow region to the free boundary) and by the formulas

$$u = t \mp \sqrt{2(Q(\lambda) - \xi - h_2(\xi))}, \qquad H = (u-t)^{-1}, \tag{3.8}$$

where $Q(\lambda)$ is a desired function in the backward-flow region; the plus sign is taken for $0 \leqslant \lambda \leqslant \mu$, and the minus sign is taken above the line $u = 0$ for $\mu \geqslant \nu \geqslant 0$ [the value of $\mu(\xi)$ is determined by the equation $Q(\mu) - \xi - h_2(\xi) = 0$)].

Integrating the function H over λ from 0 to μ, we find the height of the curve $u = 0$, and integration from μ to 0 in the region above the curve $u = 0$ determines the thickness of the backward-flow zone. Equating this quantity to the function $g(\xi)$ specified by formula (3.7), we derive the following integral equation for determination of $Q(\lambda)$:

$$2\int_0^\mu (2(Q(\lambda) - \xi - h_2(\xi)))^{-1/2}\, d\lambda = g(\xi). \tag{3.9}$$

Let us change the variables $\eta = h_2(\xi) + \xi$ and $s = Q(\lambda)$. By virtue of the conditions in $h_2(\xi)$, the function $\eta(\xi)$ is invertible for $\xi < a$. This change reduces (3.9) to the Abel equation

$$-\sqrt{2}\int_{C_m}^{\eta} \tau(s)(s-\eta)^{-1/2}\, ds = g(\xi(\eta)) = G(\eta).$$

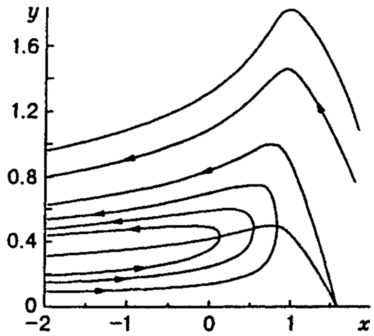

Fig. 2

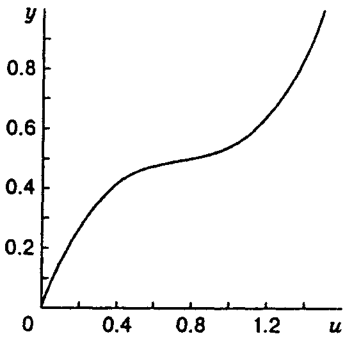

Fig. 3

Here the function $\tau(s) = -(Q'(\lambda(s)))^{-1} = \omega^{-1}$ is unknown. The solution of the Abel equation is of the form

$$\tau(s) = (\sqrt{2}\pi)^{-1} \int_{C_m}^{s} G'(\eta)(\eta - s)^{-1/2}\, d\eta.$$

Now the function $Q(\lambda)$ can be defined by solving the exact differential equation: $\tau(Q)dQ + d\lambda = 0$, $Q(0) = C_m$.

We give the example of the solution expressed in elementary functions. Let $C(\lambda) = \lambda + 3$, and, for $\eta < 3$ ($\xi < a = 3 - \sqrt{2}$), the free boundary be given as follows: $h(\xi(\eta)) = \tilde{h}(\eta) = \sqrt{2(3-\eta)}(3.5-\eta)^{-1} + \sqrt{2(4-\eta)} - \sqrt{2(3-\eta)}$. The backward-flow boundary is then determined by the function $y = G(\eta) = \sqrt{2(3-\eta)}(3.5-\eta)^{-1}$. The curve $u = 0$ is given by the equation $y = 2^{-1}G(\eta)$, the function Q is determined in the form $Q(\lambda) = 3 + 2^{-1}(1 - (1-\lambda)^{-2})$, and $\mu(\xi(\eta)) = \tilde{\mu}(\eta) = 1 - \sqrt{0.5(3.5-\eta)^{-1}}$.

The free boundary, the backward-flow region, and the curves $u = 0$ and the streamlines are shown in Fig. 2 at time $t = 0$. For $t > 0$, the flow pattern is obtained by shifting to the right by the quantity $t^2/2$ along the x axis.

Change of the Type of a System of Equations in Flow Evolution. We give a solution where Eqs. (1.3) change the type with time. We consider the invariant solution

$$u = (x - C(\lambda))t^{-1}, \qquad H = D(\lambda)t^{-1} \tag{3.10}$$

obtained after integration of the factor-system (2.8). It is of the form $u = (x - C(ty))/t$, $v = -y/t$, and $h = \text{const}/t$ in the Euler coordinate system. This solution describes vortex flow with a pressure-compressed fluid layer.

The necessary and sufficient hyperbolicity conditions of integro-differential shallow-water equations with a monotone-in-depth profile, which were given in [4], are as follows:

$$\chi^+ \neq 0, \qquad æ = \Delta \arg \chi^+(u)/\chi^-(u) = 0 \tag{3.11}$$

(the increment of the argument of the complex function χ is calculated with variation of the λ values from zero to unity with fixed t and x). The functions $\chi^{\pm}(u)$ are of the form

$$\chi^{\pm}(u(\lambda)) = 1 + \tilde{\omega}_1^{-1}(u_1 - u)^{-1} - \tilde{\omega}_0^{-1}(u_0 - u)^{-1} - \int_{u_0}^{u_1} (\tilde{\omega}^{-1})_v (v - u)^{-1}\, dv \mp \pi i (\tilde{\omega}^{-1})_u, \tag{3.12}$$

where $\tilde{\omega} = u_\lambda H^{-1} = -\omega$, and the subscripts 0 and 1 refer to the functions for $\lambda = 0$ and 1.

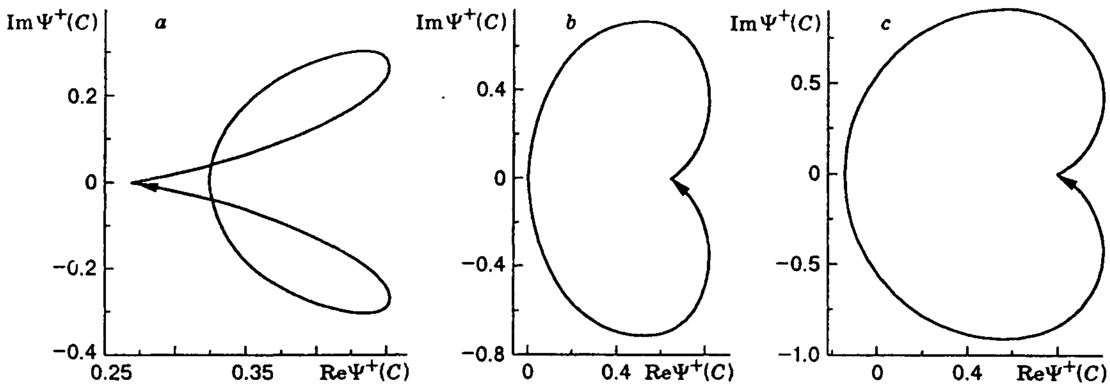

Fig. 4

Conditions (3.11) guarantee the absence of complex roots in the equation

$$1 = \int_{u_0}^{u_1} \tilde{\omega}^{-1}(u-k)^{-2}\, du,$$

which determines the velocities k of propagation of the characteristics according to [5].

We show that complex characteristic roots can appear with time in the initial solution subject to conditions (3.11). Probably, this means the loss of flow stability. Assume that $D(\lambda) \equiv 1$ in formula (3.10), and the function $C(\lambda)$ are specified unambiguously by the equation

$$C^3 + 9^{-1}C - 2^{-1} + \lambda = 0. \tag{3.13}$$

Equation (3.13) has one real and two imaginary roots for each $\lambda \in [0,1]$. Note that the derivative function $C(\lambda)$, which corresponds to vorticity, does not vanish and does not become infinite, because $\omega = C'(\lambda) = -(3C^2 + 9^{-1})^{-1}$. Owing to Eq. (3.13), we have $C(1/2) = 0$ and $-C_1 = -C(1) = C(0) = C_0 \approx 0.747$.

The velocity profile of the solution (3.10) is shown in Fig. 3 for $x = C_0$ and $t = 1$ (for other values of x and t, the diagram is obtained by shifting along the horizontal axis and by an appropriate change in the scale).

Let us verify the satisfaction of conditions (3.11) for the solution considered. In this case, the functions $\chi^\pm$ specified by formula (3.12) are of the form

$$\chi^\pm(C) = 1 + (3C_0^2 + 9^{-1})(C_0 + C)^{-1}t + (3C_0^2 + 9^{-1})(C_0 - C)^{-1}t - 12C_0t - 6Ct\ln|(C - C_0)(C + C_0)^{-1}| \pm 6\pi Cti.$$

We verify the hyperbolicity conditions in terms of the functions $\Psi^\pm$ defined by the formula

$$\Psi^\pm = (C_0^2 - C^2)\chi^\pm(C) \tag{3.14}$$

and having no poles at the points $C = \pm C_0$.

Figure 4a–c shows the diagrams of the function $\Psi^+(C)$ with variation of C from C_0 to C_1 at times $t = 0.1$, 0.239, and 0.3, respectively; the $\mathrm{Re}\Psi^+(C)$ values are plotted as the abscissa, and the $\mathrm{Im}\Psi^+(C)$ values are plotted as the ordinate (the diagrams of the function $\Psi^-(C)$ are similar, but the circumvention is performed in the opposite direction). The imaginary part of the functions $\Psi^\pm(C)$ vanishes for $C = -C_1$, 0, and C_1, and the functions themselves take the following values at these points:

$$\Psi^\pm(C_0) = \Psi^\pm(C_1) = 2C_0(3C_0^2 + 9^{-1})t > 0 \quad (t > 0), \qquad \Psi^\pm(0) = C_0^2 - 2C_0(3C_0^2 - 9^{-1})t.$$

At moment $t = t_* = 2^{-1}C_0(3C_0^2 - 9^{-1})^{-1} \approx 0.239$, the function Ψ^+ vanishes at the point $C = 0$ (see Fig. 4b), which leads to the violations of conditions (3.11). As follows from Fig. 4a, the increment of the argument of the functions $\Psi^\pm(C)$ is equal to zero, and, hence, $æ = 0$, and conditions (3.11) are satisfied for $t = 0.1$. At $t = 0.3$, based on Fig. 4c, we conclude that $\Delta \arg \Psi^+(C) = 2\pi$ and $\Delta \arg \Psi^-(C) = -2\pi$, and, hence, $æ = 4\pi$. In this case, the hyperbolicity conditions are violated. The relatively simple form of the functions $\Psi^\pm$ given by formula (3.14) allows us to make a qualitative analysis and to draw a conclusion that for any t from the interval $[t_0(t_*)$ $(0 < t_0 < t_*)$, conditions (3.11) are satisfied, and, for the solution considered, system (1.3) is hyperbolic. With $t > t_*$, the hyperbolicity conditions break down, which means the presence of the complex characteristic roots which separate from the continuous real spectrum at moment $t = t_*$. Thus, this example shows that system (1.3) can change its type in the process of flow evolution. With the fluid strip compressed, it is possible for a long-wave instability to appear in some distributions of the initial vorticity.

The author is grateful to Professor V. M. Teshukov for his attention to the work, participation in the discussion of the results, and helpful comments.

This work was supported by the International Science Foundation (Grant No. NR7000).

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